Elementary Homotopy Theory II: Pointed Homotopy

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1 Pointed Homotopy

Definition 1 *Let X, Y be based spaces. A* ***based****, or* ***pointed****,* ***homotopy*** *between based maps $f, g : X \to Y$ is a continuous function $H : X \times I \to Y$ satisfying*

1) $H(x, 0) = f(x)$, $\forall x \in X$.

2) $H(x, 1) = g(x)$, $\forall x \in X$.

3) $H(*, t) = *$, $\forall t \in I$. $\square$

Thus a based homotopy is just a homotopy through pointed maps. Clearly H factors to produce a pointed map

$$\widetilde{H} : X \times I/ * \times I \cong X \wedge I_+ \to Y \tag{1.1}$$

which satisfies the first two listed properties. Conversely, any pointed map $X \wedge I_+ \to Y$ satisfying these properties also defines a pointed homotopy. As expected, pointed homotopy is an equivalence relation on $Top_*(X, Y)$, and is compatible with composition. We use the same notation $f \simeq g$ as in the unpointed case to indicate that f, g are based homotopic. No confusion should arise, and if we need to be clear we often describe unpointed maps and homotopies as **free**.

As in the unbased case, we can also take the adjoint of a homotopy $H : X \wedge I_+ \to Y$ to view it as a based map

$$H^\# : X \to C_*(I_+, Y) \cong Y^I \tag{1.2}$$

or, if X is locally compact, an *unbased* map

$$\widetilde{H} : I \to C_*(X, Y). \tag{1.3}$$

All the notions and terminology introduced for the unpointed category are also available in the pointed category. Thus we have the idea of a pointed homotopy equivalence, a pointed contraction, etc... Many definitions go through essentially unchanged, although others are distinct. For instance, the strong deformation retractions are the same in both categories, while the (weak) deformation retractions may differ in general.

Every pointed map which is pointed null homotopic is clearly freely null homotopic, but some subtlety arises since the converse is not true. In particular there are pointed spaces which are freely contractible, but not pointed contractible. The fact that the only constant map into a pointed space is determined by its basepoint turns out to be quite significant.

Example 1.1 Recall the comb space

$$C = (I \times \{0\}) \cup (\{0\} \times I) \cup \bigcup_{n \in \mathbb{N}} \{1/n\} \times I \subseteq \mathbb{R}^2. \tag{1.4}$$

We showed when we first encountered this space that it is freely contractible. Actually, inspecting the contracting homotopy we constructed, we see that it fixes the point $(0, 0)$ at all times. Thus we may conclude that C is pointed contractible when we base it at $(0, 0)$.

On the other hand, let us consider turning C into a based space by pointing it at $(0, 1)$.

Claim: The comb C is not pointed contractible to $(0, 1)$.

From this we see explicitly how different choices of basepoints for the same underlying space give rise to different *pointed* homotopy types.

To see that our claim is true, let us assume to the contrary that C is pointed contractible to $(0, 1)$. Then there exists a contraction $F : C \times I \to C$ which satisfies $F_0 = id_C$, $F_1(C) = \{(0, 1)\}$, and $F_t(0, 1) = (0, 1)$, $\forall t \in I$.

Let $W \subseteq C$ be a neighbourhood of $(0, 1)$ such that $W \cap (I \times \{0\}) = \emptyset$. Then, given $t \in I$ we can use the continuity of F to find open neighbourhoods $U_t \subseteq W$ of $(0, 1)$ and V_t of t such that $F(U_t \times V_t) \subseteq W$. Since I is compact,we can choose finitely many such sets $V_{t_1}, \ldots, V_{t_n} \subseteq I$ whose union covers I. Taking the intersection of the corresponding sets U_{t_i}, we get an open neighbourhood $U = U_{t_1} \cap \cdots \cap U_{t_n}$ of $(0, 1)$ which is contained in W. Note that the inclusion $U \hookrightarrow W$ is based null homotopic. Indeed, the restriction $F|_{U \times I}$ is the required map.

But here we get a contradiction, since if we take a point $(x, y) \in U \setminus \{0\} \times I$, then the path $t \mapsto F(x, t)$ runs from (x, y) to $(0, 1)$ and stays inside W at all times. Clearly this can't be possible, since such a path must pass through the region $I \times \{0\}$, which is disjoint from W. $\square$

Pointed homotopy is an equivalence relation on the sets $Top_*(X, Y)$, and enjoys good properties with respect to composition. Thus can form a homotopy category of pointed spaces in much the same way as the unbased homotopy category was constructed in the exercises. We leave full details of this to the reader, who should check that the following definitions make sense.

Definition 2 *We denote by $hTop_*$ the category whose objects are pointed spaces and whose morphisms are pointed homotopy classes of maps. For pointed spaces X, Y we write*

$$[X, Y] = hTop_*(X, Y) = Top_*(X, Y)/\simeq \tag{1.5}$$

for the morphism sets in this cateory. We call $hTop_*$ the classical ***homotopy category*** *of pointed spaces.* $\square$

A morphism $[f] : X \to Y$ in $hTop_*$ is thus an equivalence class of pointed maps $X \to Y$. It is normal to identify a given map f which its homotopy class $[f]$, and write f to denote either $[f]$ or a particular representative for it. This is abuse to which we shall conform, and context will most often save us from confusion.

We can construct coproducts and products in $hTop_*$. For instance we get the former using the next lemma.

Lemma 1.1 *There are pointed homotopies $f \simeq f' : X \to Z$ and $g \simeq g' : Y \to Z$ if and only if there is a pointed homotopy $(f, g) \simeq (f', g') : X \vee Y \to Z$.*

Proof We have the homeomorphism $(X \vee Y) \wedge I_+ \cong (X \wedge I_+) \vee (Y \wedge I_+)$ which makes the following diagram commute

$$\begin{tikzcd} X \vee Y \arrow[r, equal] \arrow[d, "in_t"'] & X \vee Y \arrow[d, "in_t \vee in_t"] \\ (X \vee Y) \wedge I_+ \arrow[r, leftrightarrow, "\cong"] & (X \wedge I_+) \vee (Y \wedge I_+) \end{tikzcd} \tag{1.6}$$

where in_t generically denotes the inclusion at time $t \in I$. We conclude that a homotopy of pointed maps $X \vee Y \to Z$ is exactly a pair of pointed homotopies $X \to Z$ and $Y \to Z$. ∎

There is a similar statement for homotopy classes of maps $X \to Y \times Z$ into products of pointed spaces. Since it is identical to the corresponding unpointed statement we refrain from spelling it out. In either case, the jump from such statements to ones involving infinite products and coproducts is minor.

Corollary 1.2 *The category $hTop_*$ has all set-indexed products and coproducts.* ∎

Spelling this out in the finite case, for pointed spaces X, Y, Z there are bijections

$$\begin{array}{cc} [X \vee Y, Z] \cong [X, Z] \times [Y, Z], & [X, Y \times Z] \cong [X, Y] \times [X, Z] \\ f \mapsto (fj_X, fj_Y) & g \mapsto (pr_X g, pr_Y g) \end{array} \tag{1.7}$$

where

$$X \xrightarrow{j_Y} X \vee Y \xleftarrow{j_Y} Y \tag{1.8}$$

are the canonical inclusions, and

$$X \xleftarrow{pr_X} X \times Y \xrightarrow{pr_Y} Y \tag{1.9}$$

are the canonical projections.

In the pointed category we also have access to the smash product. This construction also respects pointed homotopy.

Lemma 1.3 *If there are homotopies $f \simeq f' : X \to Y$ and $g \simeq g' : X' \to Y'$, then there is a homotopy*

$$f \wedge g \simeq f' \wedge g' : X \wedge Y \to X' \wedge Y'. \tag{1.10}$$

Proof Let $G : f \simeq f'$ and $H : g \simeq g'$ and consider the diagram

$$\begin{tikzcd} X \times Y \times I \arrow[r, "1\times 1\times\Delta"] \arrow[d, "q\times 1"'] & X \times Y \times I \times I \arrow[r, "\cong"] & (X \times I) \times (Y \times I) \arrow[r, "G\times H"] & X' \times Y' \arrow[d, "q'"] \\ (X \wedge Y) \times I \arrow[rrr, dashed] & & & X' \wedge Y' \end{tikzcd} \tag{1.11}$$

where Δ is the diagonal and q, q' are the quotient maps. The dotted arrow represents the required homotopy. Since I is locally compact, $q \times 1$ is a quotient map, and we can check that the induced map is indeed continuous. $\blacksquare$

Remark It would be easy to write down the required homotopy directly were it not for the failure of the smash product to be associative in general. $\square$

Corollary 1.4 *The pointed homotopy type of $X \wedge Y$ depends only on those of X and Y.* $\blacksquare$

More generally, if $f : X \to X'$ and $g : Y \to Y'$ have, say, left homotopy inverses $h : X' \to X$ and $k : Y' \to Y$ (so that $hf \simeq id_X$ and $kg \simeq id_Y$), then

$$(h \wedge k)(f \wedge g) = (hf) \wedge (kg) \simeq id_X \wedge id_Y = id_{X\wedge Y} \tag{1.12}$$

which is the statement that the next diagram commutes up to homotopy

$$\begin{tikzcd} X \wedge Y \arrow[r, "f\wedge g"] \arrow[dr, equal] & X' \wedge Y' \arrow[d, "h\wedge k"] \\ & X \wedge Y. \end{tikzcd} \tag{1.13}$$

In words: if X is a homotopy retract of X' and Y is a homotopy retract of Y', then $X \wedge Y$ is a homotopy retract of $X' \wedge Y'$.

Notice one thing that we are not saying in Corollary 1.4: we are not asserting that the homotopy type of $X \wedge Y$ has any bearing on those of X or Y. Unlike the product $X \times Y$, the spaces X, Y do not retract off of $X \wedge Y$ in any natural way. There are plentiful examples of spaces X, Y for which $X \wedge Y \simeq *$, while neither X nor Y is contractible.

1.1 Cones and Paths

Every pointed space X embeds as a closed subspace of a pointed contractible space. The construction is as follows. We give $I = [0, 1]$ the basepoint 1 and define the (reduced) **cone** on X to be the space

$$CX = X \wedge I = \frac{X \times I}{X \times \{1\} \cup * \times I}. \tag{1.14}$$

This is canonically contractible by 1.4, and the map

$$j_X : X \cong X \wedge S^0 \xrightarrow{id_X \wedge in} X \wedge I = CX \tag{1.15}$$

is a closed embedding. The functorality of the construction is manifest.

As in the unpointed construction, the reduced cone has a certain weak universal property: the map j_X is the weakly initial null homotopic map out of X. We spell this out below, but first we need to discuss the dual notion, which we did not encounter in the free category.

Keeping the basepoint of $I = [0,1]$ as 1, for a pointed space Y we define

$$PY = C_*(I, X) = \{l : I \to Y \mid l(1) = *\} \tag{1.16}$$

and call it the **path space** over Y. The start point evaluation map

$$e_Y : PY \to Y, \qquad l \mapsto l(0) \tag{1.17}$$

is continuous, and is a quotient map if Y is connected and locally path-connected [2] pg. 75. Note that there is no natural map in the opposite direction in the pointed category. The construction is functorial, and a pointed map $f : X \to Y$ induces a map $Pf = f_* : PX = C_*(I, X) \to C_*(I, Y) = PY$ which makes

$$\begin{tikzcd} PX \arrow[r, "Pf"] \arrow[d, "e_X"'] & PY \arrow[d, "e_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{1.18}$$

commute.

The duality we mentioned is more than just aesthetic, and is made precise by the bijection

$$Top_*(CX, Y) \cong Top_*(X, PY) \tag{1.19}$$

which holds for any two spaces X, Y[1]. The bijection is natural in both variables, and in particular (1.19) says that the cone and path space functors are left and right adjoint.

Lemma 1.5 *For any space Y the path space PY is pointed contractible.*

Proof The contraction F_s is given by setting

$$F_s(l)(t) = l((1-s)t + s), \qquad l \in PY, s, t \in I. \tag{1.20}$$

This is clearly continuous, and for $l \in PY$ we check that

$$F_0(l) = l, \qquad F_1(l) = * \tag{1.21}$$

and that

$$F_s(l)(1) = l(1). \quad \blacksquare \tag{1.22}$$

Remark We cannot help but point out a more conceptual approach to the last proof, which unfortunately falls short of being rigorous. The idea is to notice that for each locally compact X the bijection (1.19) extends to a homeomorphism

$$C_*(CX, Y) \cong C_*(X, PY). \tag{1.23}$$

[1]This is Proposition 1.2 in *The Category of Pointed Topological Spaces*.

Then the contraction $CX \simeq *$ induces

$$C_*(CX, Y) \simeq C_*(*, Y) = *. \tag{1.24}$$

Combining the last two equations we get a contraction

$$C_*(X, PY) \simeq * \tag{1.25}$$

for each locally compact space X.

If we could remove the hypothesis of locally contractible from (1.25), then we would have the statement that any pair of maps into PY from any space are homotopic, and would get that PY is contractible by appealing to Pr. 1.5 of Lecture 1. Notice, however, that the statement that this reasoning leads to is much stronger than that given in the original Lemma. For it says that not only is PY contractible, but it is contractible in an essentially unique way. For example it say that any two contractions are track homotopic.

This is yet more motivation for us to find a way to replace the function spaces $C_*(X, Y)$ with better behaved objects. This is something we shall take up in a subsequent lecture. $\square$

Evidently a null homotopy $G : X \wedge I_+ \to Y$ of a map f takes all of $X \times \{1\}$ to the base point, and so factors to give a map

$$\hat{G} : CX \to Y \qquad \text{with} \qquad \hat{G} j_X = f. \tag{1.26}$$

Taking adjoint we can make the same argument. That is, a map $H : X \to C_*(I_+, Y) \cong C(I, Y)$ with $H(x)(0) = f(x)$ and $H(x)(1) = *$, $\forall x \in X$, restricts to give a map

$$\check{H} : X \to PY \qquad \text{with} \qquad e_Y \check{H} = f. \tag{1.27}$$

This proves:

Proposition 1.6 *A map $f : X \to Y$ is pointed null homotopic if and only if it extends over CX if and only if lifts to PY.* $\blacksquare$

We now have three distinct ways to view a null homotopy. Namely as an extension, as a lift, or as an honest homotopy:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_X"'] & Y \\ CX \arrow[ur, "\hat{F}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "*"'] \arrow[r, phantom, "\Downarrow F"] & Y \end{tikzcd} \qquad \begin{tikzcd} & PY \arrow[d, "e_Y"] \\ X \arrow[ur, "\check{F}"] \arrow[r, "f"'] & Y \end{tikzcd} \tag{1.28}$$

Notice that the diagrams on the left and right of 1.28 live in Top_*, while the diagram in the middle really does not. Rather the diagram in the middle is more suggestive of some further structure on the category. The point is that the cone and path space constructions give a way of turning the more abstract homotopy into more rigid topological data. In a sense they mediate between the homotopy category $hTop_*$ and the category Top_* itself. This is a rôle which we will see them play again, in a much more fundamentally important way, when we come to study *fiber* and *cofiber* sequences.

Example 1.2 The cone on S^n is just the disc D^{n+1}. This is obvious, but since it so clearly illustrates the situation we spell it out. The disc D^{n+1} is $*$-convex about its basepoint e_1, so is pointed contractible by means of the homotopy

$$F_t : x \mapsto (1-t)x + te_1, \qquad x \in D^{n+1}, t \in I. \tag{1.29}$$

Thus we have a null homotopy of the inclusion $f : S^n \hookrightarrow D^{n+1}$, and so an extension of f to a map $\hat{F} : CS^n \to D^{n+1}$ which is clearly a homeomorphism. $\square$

Example 1.3 The cone functor preserves pushouts. In fact the cone functor preserves all colimits. This is because it is just the functor $(-) \wedge I$, and in particular is a left adjoint according to eq. 1.19. For an example, given squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\ X \arrow[r] & X \cup_A B \end{tikzcd} \qquad \begin{tikzcd} CA \arrow[r, "Cf"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & CB \arrow[d] \\ CX \arrow[r] & C(X \cup_A B) \end{tikzcd} \tag{1.30}$$

if the left-hand diagram is a pushout, then so is the right-hand diagram, and there is thus a basepoint preserving homeomorphism

$$C(X \cup_A B) \cong CX \cup_{CA} CB. \qquad \square \tag{1.31}$$

Example 1.4 Similarly, the path space functor preserves all limits, since it is right adjoint. As an example, if

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \tag{1.32}$$

is a pullback, then there is a homeomorphism

$$P(X \times_Z Y) \cong PX \times_{PZ} PY. \qquad \square \tag{1.33}$$

Suppose $f : X \to Y$ is a null homotopic map and $F, G : f \simeq *$ are a pair of null homotopies, with corresponding extensions $\hat{F}, \hat{G} : CX \to Y$. Then a track homotopy $\psi : F \sim G$ gives rise to a homotopy $\hat{\psi}_s : \hat{F} \simeq \hat{G}$ under X. i.e. a homotopy satisfying

$$\hat{\psi}_s[x, 0] = f(x), \qquad \forall x \in X. \tag{1.34}$$

Although, of course, any two maps out of a contractible space are homotopic, the condition (1.34) is non-trivial (consider Example 1.1, for instance), and gives rise to interesting behaviour even in this situation. In the opposite direction, it's easy to see that a homotopy $H : \hat{F} \simeq \hat{G}$ under X exactly defines a track homotopy $\psi^H : F \sim G$. This reasoning leads to the following:

Proposition 1.7 *There is a one-to-one correspondence between track homotopy classes of null homotopies $f \simeq * : X \to Y$ and relative homotopy classes of extensions $\hat{f} : CX \to Y$ under X.* ∎

Of course these ideas go through in the dual picture, and we'll leave to the reader the task of filling in the details of the next statement.

Proposition 1.8 *There is a one-to-one correspondence between track homotopy classes of null-homotopies $f \simeq * : X \to Y$ and relative homotopy classes of lifts $\check{f} : X \to PY$ over Y.* ∎

1.2 Suspensions and Loops

In the last section we saw how null homotopies can be used to construct certain extensions and lifts. Moreover we saw how to understand the relative homotopy classes of the maps thus constructed through the idea of track homotopy. So, what to do with these extensions and lifts? Well, a moments reflection makes the constructions of this next section seem very natural.

Suspension We define the (reduced) **suspension** of a pointed space X by means of the pushout diagram

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[d, "j_X"'] \arrow[dr, phantom, "\lrcorner"] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X. \end{tikzcd} \tag{1.35}$$

Since the cone construction is functorial it is clear that given $f : X \to Y$ there is an induced map of pushouts

$$\Sigma f : \Sigma X \to \Sigma Y \tag{1.36}$$

which restricts to Cf on each of the cones, and hence to f on X. By reindexing the intervals in the cones we get a model for the suspension as a quotient of $X \times I$

$$\Sigma X = \frac{X \times I}{X \times \{0\} \cup X \times \{1\} \cup * \times I}. \tag{1.37}$$

As a special case of the construction we take $X = S^n$ and identify the cones with discs as in Example 1.2 to get a pushout

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D^{n+1} \arrow[d] \\ D^{n+1} \arrow[r] & \Sigma S^n. \end{tikzcd} \tag{1.38}$$

Since $S^{n+1} = D^n \cup_{S^n} D^n$ this makes clear the important homeomorphism

$$\Sigma S^n \cong S^{n+1}. \tag{1.39}$$

An explicit map for this can be found in Whitehead's book [3] pg. 107.

In fact this generalises. Take the square (1.38) and apply the functor $X \wedge (-)$ to get

$$\begin{tikzcd} X \wedge S^0 \arrow[r] \arrow[d] & X \wedge I \arrow[d] \\ X \wedge I \arrow[r] & X \wedge S^1. \end{tikzcd} \tag{1.40}$$

Lemma 1.9 *For any pointed space X, the square* (1.40) *is a pushout.*

Proof Let Z be a pointed space and consider the squares

$$\begin{tikzcd} Top_*(X \wedge S^1, Z) \arrow[r] \arrow[d] & Top_*(X \wedge I, Z) \arrow[d] \\ Top_*(X \wedge I, Z) \arrow[r] & Top_*(X \wedge S^1, Z) \end{tikzcd} \qquad \begin{tikzcd} Top_*(X, C_*(S^1, Z)) \arrow[r] \arrow[d] & Top_*(X, PZ) \arrow[d] \\ Top_*(X, PZ) \arrow[r] & Top_*(X, Z) \end{tikzcd}$$

which are obtained from each other by adjunction. Then (1.40) is a pushout if and only if for all Z the left-hand square is a pullback, if and only if for all Z the right-hand square is a pullback, if and only if for all Z, the square

$$\begin{tikzcd} C_*(S^1, Z) \arrow[r] \arrow[d] & PZ \arrow[d, "e_Z"] \\ PZ \arrow[r, "e_Z"] & Z \end{tikzcd} \tag{1.41}$$

is a pullback. But this is easy to see. ∎

Remark The lemma is trivially true when X is locally compact. □

With the lemma in hand we may compare (1.40) with the defining square (1.35). When we recall how the inclusion j_X was defined in (1.15) we see that they are identical, so

$$\Sigma X \cong X \wedge S^1. \tag{1.42}$$

Using the model (1.37) its easy to see that this homeomorphism is induced by factoring the composite $X \times I \to X \times S^1 \to X \wedge S^1$ over the quotient $X \times I \to \Sigma X$. In fact this makes apparent that (1.42) is natural in X.

Corollary 1.10 *The homotopy type of ΣX depends on X only through its homotopy type.*

Proof This follows by writing $\Sigma X \cong X \wedge S^1$ and applying Corollary 1.4. ∎

In particular, if $X \simeq Y$, then $\Sigma X \simeq \Sigma Y$. We will be quick to point out that the converse is not true. We will encounter spaces X, Y for which $\Sigma X \simeq \Sigma Y$ but $X \not\simeq Y$, and spaces Z for which $\Sigma Z \simeq *$ while $Z \not\simeq *$. Nevertheless, suspension is an important tool in homotopy theory. For instance, the suspension isomorphisms in cohomology

$$H^n X \cong H^{n+1} \Sigma X \tag{1.43}$$

mean that if you are only interested in the homotopy type of X in relation to cohomological information, then it will suffice to study instead ΣX. This is generally easier to do because, as we will see, the homotopy set $[\Sigma X, Y]$ of maps out of a suspension has a group structure, which greatly facilitates its computation.

The homeomorphism 1.42 can be iterated. Write $\Sigma^0 X = X$ and for $n \geq 1$ set $\Sigma^n X = \Sigma(\Sigma^{n-1} X)$. Then

$$S^n \cong \Sigma^n S^0 \tag{1.44}$$

and

$$\Sigma^n X \cong X \wedge S^n \tag{1.45}$$

where we have used Corollary 1.7 from Lecture 2. Some care must be taken with these identifications, however. We have formed all smash products with S^1 on the right, despite the fact that $X \wedge S^1 \cong S^1 \wedge X$, and there is a reason for being pedantic: although $S^1 \wedge S^1 \cong S^2$, the twist map

$$T : S^1 \wedge S^1 \xrightarrow{\cong} S^1 \wedge S^1, \qquad u \wedge v \mapsto v \wedge u \tag{1.46}$$

is *not* homotopic to the identity on S^2!

Loop Spaces The dual of the suspension is the loop space. Observations similar to Lemma 1.14 can easily be made, and we will be less explicit in this section than the last, leaving such thoughts for the reader to fill in.

Given a pointed space X we define its **loop space** ΩX as the pullback

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & PX \arrow[d, "e_X"] \\ PX \arrow[r, "e_X"] & X. \end{tikzcd} \tag{1.47}$$

Actually we encounted this space before in equation (1.41) where we obtained our preferred model for the loop space

$$\Omega X \cong C_*(S^1, X). \tag{1.48}$$

i.e. as the space of basepoint preserving maps $S^1 \to X$ in the compact-open topology. Writing $S^1 \cong I/\partial I$ we see that we can also identify ΩX as the subspace

$$\{l : I \to X \mid l(0) = * = l(1)\} \subseteq C(I, X). \tag{1.49}$$

We see using either (1.47) or (1.48) that Ω is a functor $Top_* \to Top_*$. We denote by $\Omega f : \Omega X \to \Omega Y$ its value at a map $f : X \to Y$. This is exactly the map which sends a loop l to $\Omega f(l) = f \circ l$.

Corollary 1.11 *The homotopy type of ΩX depends on X only through its homotopy type.*

Proof The statement is shown for $C(S^1, X)$ in the supplied notes on point-set topology, and the proof there goes through unchanged for the pointed function space. ∎

It is actually much easier to produce spaces $X \not\simeq Y$ for which $\Omega X \simeq \Omega Y$. Indeed, take $X = \mathbb{Z}$ and $Y = *$. Then we actually get homeomorphisms $\Omega\mathbb{Z} = * = \Omega *$. On the other hand, finding path connected examples is trickier, and this is another question which we shall take up again in future.

Combing equations (1.42) and (1.48) immediately yields the following.

Proposition 1.12 *The suspension and loop functors form an adjoint pair. That is, for any pointed spaces X, Y there is a bijection*

$$Top_*(\Sigma X, Y) \cong Top_*(X, \Omega Y). \tag{1.50}$$

which is natural in both variables. If X is locally compact, then there is a homeomorphism

$$C_*(\Sigma X, Y) \cong C_*(X, \Omega Y). \quad ∎ \tag{1.51}$$

Really. with the exposition we have given, this statement has been obvious for quite some time. Since it is so important we have spelled it out. For instance, a basic property of adjoint functors is the preservation of limits and colimits.

Corollary 1.13 *Suspension preserves all colimits. Looping preserves all limits.* ∎

Important consequences of this that we feel compelled to record explicitly are the following. Firstly, for any spaces X, Y there are homeomorphisms

$$\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y, \qquad \Omega(X \times Y) \cong \Omega X \times \Omega Y. \tag{1.52}$$

Secondly, if a commuting square

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "h"] \\ Y \arrow[r, "k"] & Z \end{tikzcd} \tag{1.53}$$

is given, then i) the left-hand diagram below is a pushout when (1.53) is a pushout, ii) the right-hand diagram below is a pullback when (1.53) is a pullback

$$\begin{tikzcd} \Sigma W \arrow[r, "\Sigma f"] \arrow[d, "\Sigma g"'] & \Sigma X \arrow[d, "\Sigma h"] \\ \Sigma Y \arrow[r, "\Sigma k"] & \Sigma Z \end{tikzcd} \qquad \begin{tikzcd} \Omega W \arrow[r, "\Omega f"] \arrow[d, "\Omega g"'] & X \arrow[d, "\Omega h"] \\ \Omega Y \arrow[r, "\Omega k"] & \Omega Z. \end{tikzcd} \tag{1.54}$$

1.3 Understanding Maps Out of a Suspension

In the last section we defined the suspension of a pointed space X as the pushout

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[d, "j_X"'] \arrow[dr, phantom, "\lrcorner"] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X. \end{tikzcd} \tag{1.55}$$

In this section we will try to understand the maps $\Sigma X \to Y$ from the suspension into a space Y. This task is much facilitated by our definition of ΣX as a pushout. In fact, with the aid of Proposition 1.6, the following is essentially tautological.

Lemma 1.14 *There is a one-to-one correspondence between maps $\Sigma X \to Y$ and pairs of null homotopies of a given map $X \to Y$.* ∎

Let us see explicitly how this works, and how two null homotopies are glued together to define a map. So, assume that $f : X \to Y$ is null homotopic and let $F, G : f \simeq *$ be a pair of null homotopies with extensions $\hat{F}, \hat{G} : CX \to Y$. The map out of the suspension defined by this data is then

$$\theta = \theta(F, f, G) : \Sigma X \to Y, \qquad x \wedge t \mapsto \begin{cases} \hat{G}(x, 1-2t) & 0 \le t \le \frac{1}{2} \\ \hat{F}(x, 2t-1) & \frac{1}{2} \le t \le 1. \end{cases} \tag{1.56}$$

Notice that one of the homotopies is necessarily turned upside down and put on the bottom cone of the suspension. While this is intuitive, it nevertheless leads to some odd behaviour, and at times an annoying need to keep track of signs and orientations.

The situation is depicted in the following diagrams

$$\begin{tikzcd} CX \arrow[d, "\hat{G}"'] & X \arrow[l] \arrow[r] \arrow[d, "f"] & CX \arrow[d, "\hat{F}"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd} \qquad \begin{tikzcd} * \arrow[d] \arrow[r, phantom, "\overset{-G}{\Rightarrow}"] & X \arrow[l] \arrow[r] \arrow[d, "f"] \arrow[r, phantom, "\overset{F}{\Rightarrow}"] & * \arrow[d] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y. \end{tikzcd} \tag{1.57}$$

On the left-hand side we take pushouts of the top and bottom rows to induces the map $\theta(F, G)$. On the other hand, this map is truly constructed using the null homotopies in the right-hand diagram. In this way we see explicitly how the information contained in the null homotopies is transferred up from X to ΣX: it matters not only that a map is null homotopic, but also the manner it which it becomes so.

Finally, let us consider a last way of viewing this information. On the right-hand side of (1.57), the vertical composite $-G + F$ is a homotopy $* \simeq *$. Intuitively this just runs the suspension coordinate up one cone and down the other. Clearly every map out of the suspension has this form. This reasoning leads us to:

Lemma 1.15 *There is a one-to-one correspondence between maps $\Sigma X \to Y$ and self homotopies of the constant map $X \xrightarrow{*} Y$.* ∎

Example 1.5
The inclusion $S^n \hookrightarrow S^{n+1}$ is null homotopic. While there is no canonical null homotopy of this map, there are a pair of preferred null homotopies. In more detail, we consider S^{n+1} with its standard embedding into $\mathbb{R}^{n+2}$, and S^n embedded in S^{n+1} at the equator. Then the inclusion $S^n \hookrightarrow S^{n+1}$ extends over both the northern and southern hemispheres to give our preferred pair of null homotopies. With reference to Proposition 1.6, let $F^+, F^- : D^{n+1} \to S^{n+1}$ be these two homotopies. Note that they are just the inclusions of the northern and southern hemispheres.

To proceed let us take for granted the fact that S^{n+1} is not contractible. Then using 1.57 we can now construct four maps $S^{n+1} \to S^{n+1}$. The first of these maps sends each hemisphere to itself and is just the identity

$$\theta(F^+, F^-)(x_0, \ldots, x_{n+1}) = (x_0, \ldots, x_{n+1}). \tag{1.58}$$

In particular, this map is not null homotopic. The next map switches the two hemispheres, and so inverts the sign of the last coordinate

$$\theta(F^-, F^+)(x_0, \ldots, x_{n+1}) = (x_0, \ldots, x_n, -x_{n+1}). \tag{1.59}$$

Again this map is not null homotopic, since it is clearly a homeomorphism.

On the other hand, the last two maps wrap both the hemispheres of their domain onto the same hemisphere of their target

$$\theta(F^+, F^+)(x_0, \ldots, x_{n+1}) = \begin{cases} (x_0, \ldots, x_n, -x_{n+1}) & -1 \leq x_{n+1} \leq 1 \\ (x_0, \ldots, x_n, x_{n+1}) & 0 \leq x_{n+1} \leq 1 \end{cases} \tag{1.60}$$

$$\theta(F^-, F^-)(x_0, \ldots, x_{n+1}) = \begin{cases} (x_0, \ldots, x_n, x_{n+1}) & -1 \leq x_{n+1} \leq 1 \\ (x_0, \ldots, x_n, -x_{n+1}) & 0 \leq x_{n+1} \leq 1 \end{cases} \tag{1.61}$$

It's not difficult to see that both these maps are null homotopic. □

References

[1] A. Hatcher, *Algebraic Topology*, Cambridge University Press, (2002). Available at http://pi.math.cornell.edu/ hatcher/AT/ATpage.html.

[2] E. Spanier, *Algebraic Topology*, McGraw-Hill, (1966).

[3] G. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, New York, (1978).